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Upper Semicontinuity of the Solution set to Parametric Vector Quasivariational Inequalities★

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Abstract. We prove the upper semicontinuity (in term of the closedness) of the solution set with respect to parameters of vector quasivariational inequalities involving multifunctions in topological vector spaces under the semicontinuity of the data, avoiding monotonicity assumptions. In particular, a new quasivariational inequality problem is proposed. Applications to quasi-complementarity problems are considered.

Key words: Closedness, Lower semicontinuity, Multifunctions, Quasi-complementarity problems, Topological vector spaces, Upper semicontinuity, Vector quasivariational inequalities

1. Introduction and Preliminaries

In the theory of variational inequalities, solution sensitivity is one of the subjects that have been most intensively studied.

Many authors used the projection method to study the continuity or Lipschitz continuity of the solution set of variational inequalities in Euclidean spaces or Hilbert spaces, e.g. Dafermos (1988), Mukherjee and Verma (1992), Noor (1992) and Yen (1995). Robinson (1995) used the so-called normal mappings and an implicit function approach to deal with the solution sensitivity of variational inequalities satisfying some smoothness assumptions.

Noor (1997), Domokos (1999), Ding and Luo (1999) and Kassay and Kolumban (2000) considered the continuity or Lipschitz continuity of the solution set of variational or quasivariational inequality problems in infinite dimensional settings. Levy (1999) analyzed the protodifferentiability of the solution mapping to a parameterized variational inequality with constraint sets being polyhedral in reflexive Banach spaces and computed the protoderivative of this mapping. [Note that this generalized derivative

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is a special kind of the contingent derivative of a multifunction, see e.g. Rockafellar and Wets (1998)].

To our knowledge all the works mentioned above and the majority of the papers in the literature examined the continuity, Lipschitz continuity of differentiability of solution mappings. Observe that to obtain a desired continuity of solution mapping the authors assumed the continuity of the same kind for the data in the variational inequality problem. Such assumptions may be too severe in practice. However, in many practical cases semicontinuity of the solution mappings may be sufficient. For instance, in the Walras–Ward model and the Arrow-Debreu–Mckenzie model of a competitive economy, an equilibrium for the economy exists provided only the upper semicontinuity with respect to the vector of good prices p of the set $Y(p)$ of the output vectors and of the set $W(p)$ of the factor prices, see e.g. Lancaster (1968). The sets $Y(p)$ and $W(p)$ in turn are the optimal solution sets of a pair of linear programming problems, which are dual to each other.

These observations have motivated our study of the upper semicontinuity of solutions of a vector quasivariational inequality problem involving multifunctions. We observe that rather few papers in the literature have dealt with this upper semicontinuity. Muu (1984) considered a scalar general variational inequality problem in the form of an equilibrium problem in a reflexive Banach space and proved the upper semicontinuity of the solution mapping in the weak topology, basing on semicontinuity, hemicontinuity and monotonicity of the mapping of the problem and employing a maximum theorem [in Berge (1968)] about the lower semicontinuity of the optimal value of a maximization problem.

Gwinner (1995) and Lignola and Morgan (1999) established certain kinds of convergence for solutions to perturbed variational inequality problems.

After submitting the previous version of this paper we observed the recent papers Chen and Zhu (2004) and Li et al. (2002) on a very relevant subject. Li et al. (2002) proved the upper semicontinuity of a kind of solution mappings of multivalued vector quasivariational inequalities in Banach spaces (with the objective space being finite dimensional). Chen and Zhu (2004) considered both upper and lower semicontinuity of the solution set of single-valued vector variational inequalities in finite dimensional spaces. Since our aim is almost the same, we will compare the results in details in Remarks 2.2 and 2.3 followed by several examples.

It should be noted that most of the mentioned works dealt with single-valued scalar variational inequalities. The vector case is considered only in Chen and Zhu (2004), Fu (2000), Giannessi (2000), Khanh and Luu (2004a, b). Multivalued problems are investigated in Chadli et al. (2000), Cubiotti (2002), Ding (1992), Kassay and Kolumban (2000) and Yao (1994). Quasivariational inequalities are dealt with only in Cubiotti (2002) and Ding and Luo (1999).

The problem under our consideration is as follows. Let X and Y be Hausdorff topological vector spaces, U be a Hausdorff topological one and $A \subset X$ be a nonempty closed and convex subset. Let $C: A \rightrightarrows 2^Y, T : U \times A \rightrightarrows 2^{L(X,Y)}$ and $K: U \times A \rightrightarrows 2^X$, be multifunctions, the values of C being closed and convex cones with nonempty interiors, where $L(X, Y)$ stands for the space of all continuous linear mappings of X into Y. Let $g : U \times A \rightrightarrows A$ be a continuous (single-valued) mapping. Consider the two problems of, for $u \in U$,

$(PQVI_u)$: finding $\bar{x} \in A \cap clK(u, \bar{x})$ such that $\forall x \in K(u, \bar{x}), \exists \bar{t} \in T(u, \bar{x})$,

$(\bar{t}, x - g(u, \bar{x})) \in Y \setminus -intC(\bar{x})$;

$(PSQVI_u)$: finding $\bar{x} \in A \cap clK(u, \bar{x})$ such that $\forall x \in K(u, \bar{x}), \forall t \in T(u, \bar{x})$,

$(t, x - g(u, \bar{x})) \in Y \setminus -intC(\bar{x})$,

where *int*(.) means the interior, *cl*(.) means the closure and (ζ, x) stands for the value of linear mapping ζ at x. We use the notations.

$S(u) := \{x \in A\text{: } x \textit{ is a solution of } (PQVI_u)\}$,
$S_1(u) := \{x \in A\text{: } x \textit{ is a solution of } (PSQVI_u)\}$.

To the best of our knowledge, problem $(PSQVI_u)$, even in the simplest case without parameters and when $K(x) \equiv K, g(x) \equiv x, C(x) \equiv C$ and $X = R^n$ was not examined, except in Khanh and Luu, (2004a, b). The existence of solutions and related issues for the case without parameters of $(PQVI_u)$ were developed in the works by Giannessi, Maugeri, De Luca, Cubiotti, Fu and Ricceri, published in the books edited by Giannessi and Maugeri (1995), Di Pillo and Giannessi (1996) and Giannessi (2000).

Recall that $F : X \rightrightarrows 2^Y$ is said to be upper semicontinuous (usc) at $x_0 \in domF := \{x \in X: F(x) \neq \emptyset\}$ if for each neighborhood N of $F(x)$, there is a neighborhood M of x such that $F(M) \subset N$. F is called usc in a set V, if it is usc at every $x \in V$. If $V = domF$, we simply say F is usc. F is termed closed at $x \in domF$, if $\forall x_\gamma \in domF: x_\gamma \to x, \forall y_\gamma \in F(x_\gamma): y_\gamma \to y, y \in F(x)$. F is called closed, if *graph* $F := \{(x, y) \in X \times Y: y \in F(x)\}$ is closed, i.e., from $(x_\alpha, y_\alpha) \in graphF$, $(x_\alpha, y_\alpha) \to (x, y)$, it follows that $(x, y) \in graphF$. Upper semicontinuity and closedness are closely related as shown in the following result

PROPOSITION 1.1. [see e.g. Konnov (2001), Proposition 2.1.1 and (Aubin and Frankowska 1990), Proposition 1.4.8].

(a) *If* $F: X \to 2^Y$ *has closed values and is usc than* F *is closed;*
(b) *If* $F(A)$ *is compact for any compact subset* A *of* $domF$ *and if* F *is closed, then* F *is usc;*
(c) *If* Y *is compact and* F *is closed then* F *is usc.*

Recall further that $F: X \rightrightarrows 2^Y$ is said to be lower semicontinuous (lsc) at $x \in domF$, if $\forall y \in F(x), \forall x_\alpha \in domF: x_\alpha \to x, \exists y_\alpha \in F(x_\alpha), y_\alpha \to y$.

2. The Closedness of $S(.)$ and $S_1(.)$

Since the closedness and the upper semicontinuity of a multifunction are closely related, for convenience, the study in this section is carried out in term of the closedness of the solution sets. Let $U_0 \subset U$ be an open subset and $u_0 \in U_0$. We assume that, for each $u \in U_0, S(u) \neq \emptyset$, and $S_1(u) \neq \emptyset$. In Example 2.1 below we will show how to check this condition.

THEOREM 2.1. *Assume*

(i) *K(.,.) is lsc in* (u_0, A) *and clK(.,.)is usc in* (u_0, A)*;*
(ii) $\forall u_\alpha \to u_0, \forall x_\alpha \to x_0, \forall y_\alpha \to y_0, \forall t_\alpha \in T(u_\alpha, x_\alpha), \exists t_\beta$ *(subnet),* $\exists t_0 \in T(u_0, x_0)$*, such that* $(t_\beta, y_\beta) \to (t_0, y_0)$*;*
(iii) $Y \backslash -intC(.)$ *is a closed multifunction.*

Then S(.) is closed at u_0*.*

Proof. Consider arbitrary nets $u_\alpha \to u_0, x_\alpha \in S(u_\alpha), x_\alpha \to x_0$. Suppose $x_0 \notin clK(u_0, x_0)$, i.e., there is a neighborhood $N(x_0)$ and a neighborhood V of $clK(u_0, x_0)$ such that

$$N(x_0) \cap V = \emptyset. \tag{1}$$

Since $clK(.,.)$ is usc at (u_0, x_0), without loss of generality we can assume that $x_\alpha \in clK(u_\alpha, x_\alpha) \subset V$ and $x_\alpha \in N(x_0)$ for every α, contradicting (1). So $x_0 \in clK(u_0, x_0)$. Again suppose to the contrary that $x_0 \notin S(u_0)$, i.e., there is $y_0 \in K(u_0, x_0)$ such that

$$(T(u_0, x_0), y_0 - g(u_0, x_0)) \subset -intC(x_0). \tag{2}$$

By the lower semicontinuity of $K(.,.)$ at (u_0, x_0), there exists $y_\alpha \in K(u_\alpha, x_\alpha)$, $y_\alpha \to y_0$. As $x_\alpha \in S(u_\alpha)$, there is $t_\alpha \in T(u_\alpha, x_\alpha)$ such that

$$(t_\alpha, y_\alpha - g(u_\alpha, x_\alpha)) \in Y \backslash -intC(x_\alpha). \tag{3}$$

By virtue of (ii), there are $t_0 \in T(u_0, x_0)$ and a subnet t_β such that

$$(t_\beta, y_\beta - g(u_\beta, x_\beta)) \to (t_0, y_0 - g(u_0, x_0)).$$

Now by (iii) we must have $(t_0, y_0 - g(u_0, x_0)) \in Y \backslash -intC(x_0)$, a contradiction with (2). Consequently, $x_0 \in S(u_0)$ and then $S(.)$ is closed at u_0. □

REMARK 2.1. Even in the special case where X and Y are normed spaces, assumption (ii) is weaker than '$\forall u_n \to u_0, \forall x_n \to x_0, \forall t_n \in T(u_n, x_n) \exists t_{n_k}$ (subsequence), $\exists t_0 \in T(u_0, x_0)$ such that $t_{n_k} \to t_0$'. Indeed, take $x_n \in R, x_n \equiv 0$ and $t_n \in R$ arbitrarily then $x_n \to x_0, t_n x_n \to t_0 x_0$, but t_n needs not to converge. In the case where X and Y are topological vector spaces, it is not convenient to define a topology in $L(X, Y)$.

THEOREM 2.2. *Assume (i) and (iii) as in Theorem 2.1 and instead of (ii)assume*

(ii') the multifunction T(.,.)is usc in (u_0, A) *and* $T(u_0, .)$ *has compact values.*

Then, S(.) is closed at u_0.

Proof. Consider arbitrary nets $u_\alpha \to u_0, x_\alpha \in S(u_\alpha), x_\alpha \to x_0$. Similarly as above, $x_0 \in clK(u_0, x_0)$. Suppose to the contrary that $x_0 \notin S(u_0)$, i.e., there is $y_0 \in K(u_0, x_0)$ satisfying (2). Since $K(.,.)$ is lsc at (u_0, x_0), one has $y_\alpha \in K(u_\alpha, x_\alpha)$ such that $y_\alpha \to y_0$. Since $x_\alpha \in S(u_\alpha), \exists t_\alpha \in T(u_\alpha, x_\alpha)$ such that (3) holds. By the (ii') there are a subnet t_β and $t_0 \in T(u_0, x_0)$ such that $t_\beta \to t_0$. Thus $(t_\beta, y_\beta - g(u_\beta, x_\beta)) \to (t_0, y_0 - g(u_0, x_0))$. Assumption (iii) and (3) together imply that $(t_0, y_0 - g(u_0, x_0)) \in Y \backslash -intC(x_0)$, which is impossible, due to (2). So, $x_0 \in S(u_0)$. □

Passing to $(PSQVI_u)$ we have

THEOREM 2.3. *Assume (i) and (iii) as in Theorem 2.1 and instead of (ii) assume*

(ii") the multifunction (T(.,.),.) is lsc in (u_0, A, A).

Then, S_1*(.) is closed at* u_0.

Proof. Consider arbitrary nets $u_\alpha \to u_0, x_\alpha \in S_1(u_\alpha), x_\alpha \to x_0$. Similarly as above, $x_0 \in clK(u_0, x_0)$. Suppose to the contrary that $x_0 \notin S_1(u_0)$, i.e., $\exists y_0 \in K(x_0, u_0), \exists t_0 \in T(u_0, x_0)$,

$$(t_0, y_0 - g(u_0, x_0)) \in -intC(x_0). \tag{4}$$

Since $K(.,.)$ is lsc at (u_0, x_0), one has $y_\alpha \in K(u_\alpha, x_\alpha)$ such that $y_\alpha \to y_0$. Since $(T(.,.),.)$ is lsc at (u_0, x_0, y_0), there is a net $t_\alpha \in T(u_\alpha, x_\alpha)$ such that

$$(t_\alpha, y_\alpha - g(u_\alpha, x_\alpha)) \to (t_0, y_0 - g(u_0, x_0)).$$

One has $(t_\alpha, y_\alpha - g(u_\alpha, x_\alpha)) \in Y \backslash -intC(x_\alpha)$, for $x_\alpha \in S_1(u_\alpha)$. Since

$Y\backslash -intC(.)$ is closed,

$$(t_0, y_0 - g(u_0, x_0)) \in Y\backslash -intC(x_0),$$

which is impossible due to (4). So $x_0 \in S_1(u_0)$, and then $S_1(.)$ is closed at u_0. □

EXAMPLE 2.1. Consider $(PQVI_u)$ with $X = Y = U = R$, $A = [0, 1]$, $C(x) \equiv R_+$, $g(u, x) \equiv x$ and, $\forall u \in U$,

$$K(u, x) = \left[0, \frac{u+x}{2}\right), \quad x \in [0, 1],$$

$$T(u, x) = \begin{cases} R \quad if \ x = 0, \quad or \quad x = 1, \\ [-u, u] \quad if \ x \in \left(0, \frac{1}{2}\right], \\ \{u^2 + x^2\} \quad if \ x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

To study the solution sensitivity of this problem, the theorems of Berge (1968) and Li et al. (2002) are not applicable. (All the other sensitivity results known to us cannot be used as we are faced with a quasivariational inequality). Now we apply Theorem 2.1 for an arbitrary u_0 in U_0, say $U_0 =$ (0,1). Assumptions (i) and (iii) are clearly satisfied. To check (ii) take arbitrarily convergent sequences $u_n \to u_0, x_n \to x_0, x_n \neq x_0, y_n \to y_0$ and points $t_n \in T(u_n, x_n)$. If $0 < x_0 \leqslant \frac{1}{2}$ and $0 < x_n \leqslant \frac{1}{2}$, then $t_n \in [-u_n, u_n] \subset [-u_0 - \delta, u_0 + \delta]$, with a fixed δ for sufficiently large n. Hence, there are $t_0 \in [-u_0, u_0]$ and subsequence $t_{n_k} \to t_0$. Therefore, $t_{n_k} y_{n_k} \to t_0 y_0$. If $x_n > \frac{1}{2}$ for every n, then $t_n = u_n^2 + x_n^2 \to u_0^2 + x_0^2 =: t_0$ and again $t_n y_n \to t_0 y_0$. If $\frac{1}{2} < x_0 < 1$, then (for large n) $T(u_n, x_n) = \{u_n^2 + x_n^2\}$ and, as above, (ii) is fulfilled. If $x_0 = 0$, since $x_n \neq x_0$, the situation is similar to the case where $0 < x_0 \leqslant \frac{1}{2}, 0 < x_n \leqslant \frac{1}{2}$, while if $x_0 = 1$, the argument is similar to that for the case where $\frac{1}{2} < x_0 < 1$. So (ii) is satisfied and $S(.)$ is closed. Since $A = [0, 1]$ is compact, by Proposition 1.1 (c), $S(.)$ is also usc. Note that Theorem 2.2 cannot be applied since $T(u_0, 0)$ and $T(u_0, 1)$ are not compact.

Using Theorem 2.1 of Khanh and Luu (2004a ,b) we can show that $S(u) \neq \emptyset, \forall u \in U_0 = (0, 1)$.

EXAMPLE 2.2. Consider $(PSQVI_u)$ with $X, Y, A, C(x)$, $g(u, x)$ and $K(u, x)$ as in Example 2.1, and, for each $u \in U := (0, 1)$,

$$T(u, x) = \begin{cases} \left(u, \frac{2}{u}\right) \ if \ x \in \left[0, \frac{1}{2}\right), \\ \left\{u + x^2\right\} if \ x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider any point $u_0 \in (0, 1)$.

Then, all assumptions of Theorem 2.3 are clearly fulfilled. Thus, $S_1(.)$ is closed at u_0. Note that all assumption of Theorem 2.2 in Khanh and Luu (2004a ,b) are also satisfied $\forall u \in U$. So $S_1(u) \neq \emptyset, \forall u \in U$.

REMARK 2.2. When applied to the special case of our problems $(PQVI_u)$ and $(PSQVI_u)$, which is considered in Chen and Zhu (2004), Theorems 2.1–2.3 have weaker assumptions than that of Theorem 2.1 in Chen and Zhu (2004), since the values of $K(.,.)$ are allowed to be nonconvex and non-compact, $K(.,.)$ needs not to be uniformly compact near u_0 and $T(\cdot, u_0)$ needs not to be pseudomonotone.

REMARK 2.3. Unlike the single-valued case, for the multivalued quasi-variational inequalities, three problems arise naturally. Beside $(PQVI_u)$ and $(PSQVI_u)$, the third problem is

$$(P_u)\text{: finding } \bar{x} \in A \cap clK(u, \bar{x}) \text{ such that } \exists \bar{t} \in T(u, \bar{x}), \forall x \in K(u, \bar{x}),$$
$$(\bar{t}, x - g(u, \bar{x})) \in Y \setminus -intC(\bar{x});$$

The solution existence of this problem was considered (for variational or quasivariational inequalities with a variety of settings) e.g. in Chadli et al. (2000), Cubiotti (2002), Ding (1992) and Yao (1994). The continuity of the solution mapping was analyzed in Ding and Luo (1999). Denoting

$$I^*(u) := \{x \in A : x \ is \ a \ solution \ of \ (P_u)\},$$

one clearly has

$$S_1(u) \subset I^*(u) \subset S(u). \tag{5}$$

In Li et al. (2002), the following extraordinary solution multifunction was proved to be closed

$$I(u) := \{\bar{x} \in A : \exists \bar{t} \in T(u, \bar{x}), \ \forall x \in K(u, \bar{x}),$$
$$(\bar{t}, x - g(u, \bar{x})) \in Y \setminus -intC(\bar{x})\}.$$

Note that the difference of $I(u)$ from $I^*(u)$ is that $\bar{x} \in A$, not $\bar{x} \in A \cap clK(u, \bar{x})$ and that $I^*(u) \subset I(u)$. [In fact the notation $I(u)$ in Li et al. (2002) is for the set of $(\bar{x}, \bar{t})$ in the product space].

The following example shows that the inclusions in (5) may be proper and the closedness of the three multifunctions may be different.

EXAMPLE 2.3. Let $X, Y, A, U, C(x)$ and $g(u,x)$ be as Example 2.1. For all $u \in U$, let $K(u,x) = [1,0]$ and

$$T(u,x) = \begin{cases} \{1\} & if\ x = 0 \quad or \quad x = \frac{1}{2}, \\ \{-1,1\} & if\ x = \frac{n-1}{2n}, \quad n = 2,3,\ldots or \quad x = 1, \\ \{-1\} & otherwise. \end{cases}$$

Then, it is not hard to see that $S_1(u) = \{0\}$, $I^*(u) = \{0,1\}$ and $S(u) = \{0,1\} \cup \{\frac{n-1}{2n} : n = 2,3,\ldots\}$. So, $S_1(.)$ and $I^*(.)$ are closed but $S(.)$ is not closed. Moreover, one can check directly that assumption (ii) of Theorem 2.1 and (ii′) of Theorem 2.2 are not satisfied, respectively. However, assumption (ii″) is also violated. But $S_1(.)$ is closed. So, Theorem 2.3 gives a sufficient condition, which is not necessary.

EXAMPLE 2.4. Let all the data be as in Example 2.3, except $T(u,x)$ which is defined by

$$T(u,x) = \begin{cases} \{-1,1\} & if\ x = \frac{1}{2}, \\ \{0\} & otherwise. \end{cases}$$

Then, $S_1(u) = I^*(u) = [0,1] \setminus \left\{\frac{1}{2}\right\}$ and $S(u) = [0,1]$. Hence, $S(.)$ is closed, but $S_1(.)$ and $I^*(.)$ are not closed.

The following example clarifies that $I^*(.)$ may be properly included in $I(u)$.

EXAMPLE 2.5. Let $X, Y, A, U, C(x)$ and $g(u,x)$ be as in Example 2.1. For all $u \in U$, let $K(u,x) = [0, \frac{x+u}{4}]$ and $T(u,x) = \{-1,1\}$. Then, $I^*(u) = \{0\} \cup \left\{\frac{u}{3}\right\}$ and $I(u) = \{0\} \cup [\frac{u}{3}, 1]$.

The assumptions of Theorem 3.2, in Li et al. (2002) are similar to that of our Theorem 2.2 but $clK(.,.)$ needs not to be usc, since the elements $\bar{x}$ of $I(u)$ need not to belong to $clK(u,\bar{x})$. The example below highlights that this upper semicontinuity is crucial for our three theorems to hold.

EXAMPLE 2.6. Let $X, Y, A, U, C(x)$ and $g(u,x)$ be as in Example 2.1. For all $(u,x) \in U \times A$, let

$$K(u,x) = \begin{cases} \{0\} & if\ u = 0, \\ \left[0, \frac{1}{2} + \frac{x+u}{4}\right] & otherwise, \end{cases}$$

$$T(u,x) = \{-1,1\}.$$

Then, $clK(.,.)$ is not usc in (u_0, A). By direct computation one has

$$S(u)=\begin{cases}\{0\} & if\ u=0,\\ \left[0,\frac{2+u}{3}\right] & otherwise,\end{cases}$$

$$S_1(u)=\begin{cases}\{0\} & if\ u=0,\\ \emptyset & otherwise.\end{cases}$$

3. Applications to Quasi-complementarity Problems

Let X, U, A, T and K be as in the definition of problem $(PSQVI_u)$. As for the other data of the problem we restrict ourselves to the special case $Y=R, C(x)\equiv x$ and $g(u,x)=x, \forall u\in U$, Let $H: U\times A\rightrightarrows 2^X$ be a multifunction. We define a quasi-complementarity problem with parameters as follows, for $u\in U$,

(PQC_u): find $\bar{x}\in A$ such that $\forall\bar{h}\in A\cap H(u,\bar{x}), \forall t\in(-A^0)\cap T(u,\bar{x})$,
$$\langle t,\bar{h}\rangle=0.$$

Here, for $A\subset X$, the polar set A^0 is defined by

$$A^0:=\{f\in X^*:\langle f,a\rangle\leqslant 1,\ \ \forall a\in A\}.$$

If A is a cone, then it is clear that A^0 coincides with the negative conjugate cone

$$-A^*=\{f\in X^*:\langle f,a\rangle\leqslant 0, \forall a\in A\}.$$

If $H(u,x)\equiv A$ and $T(u,x)=t(x)$ begin a single-valued mapping, then (PQC_u) reduces to the complementarity problem introduced by Karamardian (1971) and has been intensively investigated. Another quasi-complementarity problem involving multifunctions was introduced in Fu (2000) with $\forall\bar{h}, \forall t$ in (PQC_u) replaced by $\exists\bar{h}, \exists t$. In Cubiotti (2002) a quasi-complementarity problem with single-valued mappings was studied. Complementarity problems are closely related to variational inequalties due to the following

LEMMA 3.1 [Karamardian (1971)]. *Let X be a Hausdorff topological vector space and $A\subset X$ be a closed and convex cone. Then $\bar{x}\in A, \bar{t}\in X^*$ and $\langle\bar{t},x-\bar{x}\rangle\geqslant 0,\ \forall x\in A$ if and only if $\bar{x}\in A, \bar{t}\in A^*$ and $\langle\bar{t},\bar{x}\rangle=0$.*

Let T' be the multifunction defined by $T'(u,x)=(-A^0)\cap T(u,x)$ and $(PSQVI'_u)$ be $(PSQVI_u)$ with T replaced by T'.

THEOREM 3.2. (i) *Let the multifunction K in $(PSQVI'_u)$ and H in (PQC_u) satisfy the relation, $\forall(u,x)\in U\times A$,*

$$K(u,x) = x - A \cap H(u,x) + A \tag{6}$$

and A be a closed and convex cone. Then $\bar{x}$ is a solution of $(PSQVI'_u)$ if and only if $\bar{x}$ is a solution of (PQC_u).

(ii) *If, beside the assumptions in (i),H(.,.) has closed values and is continuous in (u_0, A) and $(T(.,.),.)$ is wlsc in (u_0, A, A), then the solution multifunction $W(.)$ defined by*

$$W(u) := \{x \in A : x \textit{ is a solution of } (PQC_u)\}$$

is closed at u_0.

Proof. (i) Assume that $\bar{x}$ is a solution of (PQC_u), Then $\forall x \in K(u,\bar{x}), \exists \bar{h} \in A \cap H(u,\bar{x}), \exists a \in A, x = \bar{x} - \bar{h} + a$. Hence, taking Lemma 3.1 into account we have, $\forall t \in A^* \cap T(u,\bar{x}), \langle t, x - \bar{x}\rangle = \langle t, a - \bar{h}\rangle \geqslant 0$, i.e., $\bar{x}$ is a solution of $(PSQVI'_u)$.

Conversely, assume that $\bar{x}$ is a solution of $(PSQVI'_u)$. Since $\forall \bar{h} \in A \cap H(u,\bar{x}), \forall a \in A, x : \bar{x} - \bar{h} + a \in K(u,\bar{x})$, one has, for each $t \in T'(u,\bar{x})$,

$$0 \leqslant \langle t, (\bar{x} - \bar{h} + a) - \bar{x}\rangle = \langle t, a - \bar{h}\rangle.$$

Invoking to Lemma 3.1 one gets $\langle t, \bar{h}\rangle = 0$, i.e., $\bar{x} \in W(u)$.

(ii) Clearly all assumptions of Theorem 2.3 are fulfilled and then this theorem together with (i) establish the closedness of $W(.)$ at u_o. □

Note that, for a given $K(.,.)$, the multifunction $H(.,.)$ satisfying (6) is not unique, i.e., we have a family of (PQC_u) corresponding to a given $(PSQVI'_u)$. However, according to Theorem 3.2 (i), all the problems of the family have a common solution set being the solutions set of $(PSQVI'_u)$.

4. Conclusions

Parametric multivalued vector quasivariational inequalities have been considered in Hausdorff topological vector spaces. The upper semicontinuity of the solution set of these problems was established. Examples were provided to explain some advantages of our results and to compare the solution sets of the mentioned different problems. A direct application to corresponding quasi-complementarity problems was also presented.

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